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Sheets

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In studying the question of the stability of vortex trails with staggered vortices when there occurs "group displacements" (for example, alternating) or in finding the velocity of asymmetric trails we always encounter the effect of oblique flow of trails. This means that the axis of the vortex system, while maintaining a direction parallel to the general motion, with the **mx** course of time, moves infinitely far from the axis of symmetry of the streamlines of a body (cylinder). In the case of staggered trails, displaced alternately, we obtain for the law of motion, for example for vortices of the upper chain with even index, the expression [1,2]

$$\xi_{2i}'(t) = \frac{\xi_0' + \xi_1'}{2} + \frac{\lambda}{2}\left\{(\eta_0' + \eta_1') - (\eta_0'' + \eta_1'')\right\}t$$

$$+ \frac{1}{2}\left[e^{\mu t}(a'\cos \nu t + b''\sin \nu t)\right.$$

$$\left. + e^{-\mu t}(a'\cos \nu t + b''\sin \nu t)\right] \tag{1}$$

$$(i = 0,1,2\ldots, \mu = 0 \equiv \chi = h/l \approx 0.281)$$

In the case of asymmetric trails, oblique flow arises from the fact that for the secondary component V_A of the velocity $\vec{W}$ of any vortex has the value

$$V_A = -\frac{\Gamma}{4l}\frac{\sin 2\lambda\pi}{\cosh 2\chi\pi - \cos 2\lambda\pi}; \quad U_A = \frac{\Gamma}{4l}\frac{\sinh 2\chi\pi}{\cosh 2\chi\pi - \cos 2\lambda\pi} \tag{2}$$

which for $\lambda \neq 0, \frac{1}{2}$ also differs from zero. On the basis of the

above, it is possible to assert that not only asymmetric trails
but also staggered vortex trails even when they satisfy the Karman
condition [3]

$$\sinh \varkappa \pi = 1 \tag{3}$$

$\varkappa = h/\iota$ are unstable in so far as it is possible for us to find such
special displacements which produce oblique motion of it. Actually,
in the specified cases the general definition of stability is not
fulfilled, according to which it is necessary in order that "for
arbitrarily small displacements of all or some vortices in the initial
moment of time, all the vortices, with the lapse of time, remain near
that position which they would have had if they moved without also
undergoing displacement". Departing from this general definition,
let us assume, say, the possibility of oblique travel of the vortex
system, for example, of asymmetric trails. We must study its stab-
ility or instability with respect to a more narrow definition of
stability which was formulated by Kochin [3] as: "Let us call the
vortex system stable if for any positive number ε , as small as
desired, it is possible to choose such a positive number δ that
with the vortex displacements not exceeding the quantity δ at
the initial moment of time, the distance between any two vortices
in all the time of the motion of the system will be different from
the distance between these vortices in the undisturbed state not
larger than ε ."

Thus, we must consider the question of the stability or instab-
ility of asymmetric vortex trails in the light of the above definition

since as regards staggered trails we established earlier [1,2] that they are stable by condition (3) even for displacements of second order. However, as deduced by Kochin [3] following the method of Liapunov, staggered trails with the attraction displacement of fourth order proved unstable in spite of condition (3) being fulfilled. The analogy between the behavior of asymmetric and staggered trails is established here according to Kochin's method; what is more valuable, in the following work we apply asymmetric trails as transitions to Karman staggered trails when they are "made stable" in the latter stages of the motion of two oscillating vortex chains. The complete analogy, which we establish in our generalized study, gives us the right to consider asymmetric trails for conditions of stability different from (3) such which, just the same as the Karman trail are "less unstable". Therefore, it is possible to expect that an a-symmetric trail in spite of a brief oblique flow transforms without collapse into a trail with vortices in staggered arrangement.

Now we start from the alternate displacement of the vortex trails with asymmetric vortex arrangement whose motion is governed by the system of differential is equations [3] :

$$\frac{d\overline{z_1}'}{dt} = \frac{\Gamma}{4\iota i}\left\{\cot\frac{\pi}{2\iota}(z_1'-z_2') - \cot\frac{\pi}{2\iota}(z_1'-z_1'') - \cot\frac{\pi}{2\iota}(z_1'-z_2'')\right\}$$

$$\frac{d\overline{z_2}'}{dt} = \frac{\Gamma}{4\iota i}\left\{\cot\frac{\pi}{2\iota}(z_2'-z_1') - \cot\frac{\pi}{2\iota}(z_2'-z_1'') - \cot\frac{\pi}{2\iota}(z_2'-z_2'')\right\}$$

$$\frac{d\overline{z_1}''}{dt} = \frac{\Gamma}{4\iota i}\left\{\cot\frac{\pi}{2\iota}(z_1''-z_1') - \cot\frac{\pi}{2\iota}(z_1''-z_2') - \cot\frac{\pi}{2\iota}(z_1''-z_2'')\right\} \tag{4}$$

$$\frac{d\overline{z_2}''}{dt} = \frac{\Gamma}{4\iota i}\left\{\cot\frac{\pi}{2\iota}(z_2''-z_1') - \cot\frac{\pi}{2\iota}(z_2''-z_2') - \cot\frac{\pi}{2\iota}(z_2''-z_1'')\right\}$$

while the non-displaced vortex chains move with the velocity (2).

The asymmetric initial position of the vortex we fix by the formulas

$$z_{10}' = d + \tfrac{1}{2}ih; \quad z_{10}'' = -d - \tfrac{1}{2}ih; \quad z_{20}' = l + d + \tfrac{1}{2}ih; \quad z_{20}'' = l - d - \tfrac{1}{2}ih \quad (5)$$

For brevity, let us put $\dfrac{\Gamma \pi}{8 l^2} t = \tau$ and introduce corresponding

with this for the k-th affix of the alternate displaced vortex the expression

$$z_k = \frac{\Gamma}{2l}(\overline{U}_A - i\overline{V}_A)t + z_{ko} + \frac{2i}{\pi}\zeta \quad (6)$$

in which $\overline{U}_A$ and $\overline{V}_A$ are, with the accuracy of a known factor, the components of (2); and ζ is the affix of the displacement of the k-th vortex; the system (4) assumes the form

$$\frac{d\bar{\zeta}_1}{d\tau} = i\left\{\tan(\zeta_1' - \zeta_1'') + \cot(\zeta_1' - \zeta_2' + \lambda\pi + \tfrac{1}{2}\kappa\pi)\right.$$
$$\left. + \cot(\zeta_1' - \zeta_2'' + \lambda\pi + \tfrac{1}{2}\kappa\pi - \tfrac{1}{2}\pi) - 2(\overline{U}_A - i\overline{V}_A)\right\}$$

$$\frac{d\bar{\zeta}_1'}{d\tau} = i\left\{\tan(\zeta_2'' - \zeta_1') + \cot(\zeta_1' - \zeta_2' + \lambda\pi + \tfrac{1}{2}\kappa\pi)\right.$$
$$\left. + \cot(\zeta_1'' - \zeta_2' + \lambda\pi + \tfrac{1}{2}\kappa\pi + \tfrac{1}{2}\pi) - 2(\overline{U}_A - i V_A)\right\} \quad (7)$$

$$\frac{d\bar{\zeta}_1''}{d\tau} = i\left\{\tan(\zeta_1'' - \zeta_1') + \cot(\zeta_1'' - \zeta_2'' + \lambda\pi + \tfrac{1}{2}\kappa\pi)\right.$$
$$\left. + \cot(\zeta_1'' - \zeta_2' + \lambda\pi + \tfrac{1}{2}\kappa\pi + \tfrac{1}{2}\pi) - 2(\overline{U}_A - i\overline{V}_A)\right\}$$

$$\frac{d\bar{\zeta}_2''}{d\tau} = i\left\{\tan(\zeta_1' - \zeta_2'') + \cot(\zeta_1'' - \zeta_2'' + \lambda\pi + \tfrac{1}{2}\kappa\pi)\right.$$
$$\left. + \cot(\zeta_1' - \zeta_2'' + \lambda\pi + \tfrac{1}{2}\kappa\pi - \tfrac{1}{2}\pi) - 2(\overline{U}_A - i V_A)\right\}$$

The integral of this system is $\zeta_1' - \zeta_2' + \zeta_1'' - \zeta_2'' = C$

and the requirement of single-valuedness of the l vortex of basis

form as a consequence of which $C = 0$, leads us to the relation

$$\zeta_2' - \zeta_1' = \zeta_1'' - \zeta_2'' \quad \text{or} \quad \zeta_2'' - \zeta_1' = \zeta_1'' - \zeta_2'.$$

In order to obtain the most general solution of the system (7)

we consider the obtained equation relating to both series and we put

$$2(\mathcal{F}_2' - \mathcal{F}_1') = \alpha = 2(\mathcal{F}_1'' - \mathcal{F}_2'')$$

$$2(\mathcal{F}_2'' - \mathcal{F}_1') = \beta = 2(\mathcal{F}_1'' - \mathcal{F}_2')$$

Then (7) is transformed into

$$\frac{d\bar{\alpha}}{d\tau} = 4i \sin\beta \left(\frac{1}{\cos\alpha + \cos\beta} - \frac{1}{\cos\beta + a} \right)$$

$$\frac{d\bar{\beta}}{d\tau} = 4i \sin\alpha \left(\frac{1}{\cos\alpha + \cos\beta} - \frac{1}{\cos\beta - a} \right)$$

(8)

where $a = \cos(\lambda\pi + i\varkappa\pi)$. For $\lambda = \frac{1}{2}$, it is immediately evident that the system reduces to the Kochin [3] system for stability, according to formula (3) of staggered Karman trails. In order to study the system (8) let us retain in it only terms which are linear with respect to α and β. In the same way the system (8) becomes

$$\frac{d\bar{\alpha}}{d\tau} = -2i \frac{1-a}{1+a} \beta \quad ; \quad \frac{d\bar{\beta}}{d\tau} = -2i \frac{1+a}{1-a} \alpha$$

(9)

and the conjugate

$$\frac{d\alpha}{d\tau} = 2i \frac{1-\bar{a}}{1+\bar{a}} \bar{\beta} \quad ; \quad \frac{d\beta}{d\tau} = 2i \frac{1+\bar{a}}{1-\bar{a}} \bar{\alpha}$$

(10)

The systems (9) and (10) are satisfied by the particular solutions $\alpha = M e^{\omega t}$; $\bar{\alpha} = N e^{\omega t}$; $\beta = R e^{\omega t}$; $\bar{\beta} = S e^{\omega t}$ which lead us to the characteristic system of algebraic equations

$$M\omega - LS = 0; \quad R\omega - \bar{K}N = 0; \quad N\omega + LR = 0; \quad S\omega + KM = 0 \quad (11)$$

homogeneous with respect to M, N, R and S. Here we let

$$K = -2i \frac{1+a}{1-a} \quad ; \quad \bar{K} = 2i \frac{1+\bar{a}}{1-\bar{a}} \quad ; \quad L = -2i \frac{1-a}{1+a} \quad ; \quad \bar{L} = 2i \frac{1-\bar{a}}{1+\bar{a}} \quad (12)$$

Elimination of M, N, R, and S from (11) leads us to the equation

$$\omega^4 + (L\bar{K} + \bar{L}K)\omega^2 + L\bar{L}K\bar{K} = 0 \quad (13)$$

The roots of the corresponding quadratic equation if $\varepsilon = \omega^2$ are given by the expressions

$$\varepsilon_1 = 4 \frac{(1+a)(1-\bar{a})}{(1-a)(1+\bar{a})} = 4\left(\frac{1 - |a| - 2i \sin \lambda\pi \sinh \kappa\pi}{1 - |a| + 2i \sin \lambda\pi \sinh \kappa\pi}\right)$$

$$\varepsilon_2 = 4 \frac{(1-a)(1+\bar{a})}{(1+a)(1-\bar{a})} = 4\left(\frac{1 - |a| + 2i \sin \lambda\pi \sinh \kappa\pi}{1 - |a| - 2i \sin \lambda\pi \sinh \kappa\pi}\right) \tag{14}$$

since $a - \bar{a} = -2i \sin \lambda\pi \sinh \kappa\pi = -Qi$. For the roots of ω, letting $R^2 = (1 - |a|)^2 + Q^2$, we obtain

$$\omega_{1,2,3,4} = \pm \frac{2}{R}(1 - |a| \pm Qi) \tag{15}$$

If $(1 - |a|) \neq 0$, then two of the roots of (13) will have positive real part as a consequence of which the solution $\alpha = 0$, $\beta = 0$ will be an unstable solution of the systems (9) and (10). If, however $|a| = 1$, or what is the same

$$|\cos(\lambda\pi + i\kappa\pi)| = 1 \tag{16}$$

then we again find the condition

$$\sinh \kappa\pi = \sin \lambda\pi \tag{17}$$

which is necessary for the stability of asymmetric vortex trails for a finite displacement of all vortices.

But now equation (13) has two pure imaginary double roots, since from R and (16) it follows that $R = Q$. Therefore (15) is reduced to $\omega = \pm 2i$. In the same way, for asymmetric trails we encounter the case when the first approximation is insufficient to maintain that the solution $\alpha = 0$, $\beta = 0$ is stable. However, the obtained necessary condition of stability (17) from which for $\lambda = \frac{1}{2}$ the Karman condition results, gives us the right to maintain, together with Kochin [4] that "to a certain degree (17) maintains its value since it characterizes

those arrangements of vortices which possess the leasttinstability in comparison with all other vortex arraggements".

As regards later investigations having as their aim to establish by the method of Liapunov-Kochin the instability also of asymmetric trails, we only observe that for these generalized trails there is obtained functions completely analoguus to those which Kochin 3 uses:

$$F(\alpha,\beta) = \text{\textsterling} \ln \left| \frac{(\cos \alpha - a)(\cos \beta + a)}{\cos \alpha + \cos \beta} \right| \tag{18}$$

where $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ and the complex constant is determined by (8) and condition (17). The system of differential equations (8) are now replaced by the $\mathbf{t}$ equations

$$\frac{d\alpha_1}{d\tau} = \frac{\partial F}{\partial \beta_2}; \quad \frac{\partial \alpha_2}{\partial \tau} = -\frac{\partial F}{\partial \beta_1}; \quad \frac{\partial \beta_1}{\partial \tau} = \frac{\partial F}{\partial \alpha_2}; \quad \frac{\partial \beta_2}{\partial \tau} = -\frac{\partial F}{\partial \alpha_1}, \tag{19}$$

with the first integral F = constant. The rest of the reasoning corresponds completely with Kochin's investigation.

1. B. Dolapchiev: ZAMM, 17, 313 (1937)

2. B. Dolapchiev: Ann. Bulg. Acad. of Science, 57, 149, (1938)

3. N. E. Kochin: DAN, 24, #1, (1939)

4. Kochin, Kibel, Roze: Theoretical Hydrodynamics, Vol. I, Chapt. V
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